$$
\begin{aligned}
h_n &= h(nT) \\
\hat{H}_k &= H\left(\omega = \frac{2\pi k}{NT}\right)
\end{aligned}
\tag{24.436}
$$

Then, in discrete form, the integral of Equation 24.390 changes to a finite sum of the N values of the signal with $dt \to T$. Therefore,

$$
\begin{aligned}
\hat{H}_k &= \sum_{n=0}^{N-1} h_n e^{-i\frac{2\pi k}{N\cancel{T}}n\cancel{T}} T \\
&= \sum_{n=0}^{N-1} h_n e^{-i\frac{2\pi kn}{N}} T
\end{aligned}
\tag{24.437}
$$

To make the Discrete Fourier Transform independent of the sampling frequency, we define the Discrete Fourier Transform H_k such that

$$
\hat{H}_k = H_k T \tag{24.438}
$$

Therefore, the Discrete Fourier Transform (DFT) is defined as,

$$
H_k = \sum_{n=0}^{N-1} h_n e^{-i\frac{2\pi kn}{N}} \tag{24.439}
$$

Note that there is also another type discretized Fourier transform called *Discrete-Time Fourier Transform* and it should not be confused with the subject of this section which is *Discrete Fourier Transform*.

24.10.1 Inverse Discrete Fourier Transform (IDFT)

Now, to compute the inverse Discrete Fourier Transform, consider the inverse Complex Fourier Transform, Equation 24.391, and discretize it by having $d\omega \to \frac{2\omega_c}{N} = \frac{2\pi}{NT}$, then, the discretized version of Equation 24.391 will become,

$$
\begin{aligned}
h_n &= \frac{1}{2\pi} \sum_{k=0}^{N-1} \hat{H}_k e^{i\frac{2\pi k}{NT}nT} \frac{2\pi}{NT} \\
&= \frac{1}{\cancel{2\pi}} \sum_{k=0}^{N-1} H_k \cancel{T} e^{i\frac{2\pi k}{N\cancel{T}}n\cancel{T}} \frac{\cancel{2\pi}}{N\cancel{T}} \\
&= \frac{1}{N} \sum_{k=0}^{N-1} H_k e^{i\frac{2\pi kn}{N}}
\end{aligned}
\tag{24.440}
$$

Therefore, Equations 24.441 and 24.442 are the DFT and IDFT respectively,

$$H_k = \sum_{n=0}^{N-1} h_n e^{-i\frac{2\pi kn}{N}} \tag{24.441}$$

$$h_n = \frac{1}{N}\sum_{k=0}^{N-1} H_k e^{i\frac{2\pi kn}{N}} \tag{24.442}$$

It is customary to define a factor called the *twiddle factor* in the following way,

$$W_N \overset{\Delta}{=} e^{i\frac{2\pi}{N}} \tag{24.443}$$

Therefore, Equations 24.441 and 24.442 may be expressed in terms of W_N as follows,

$$H_k = \sum_{n=0}^{N-1} h_n W_N^{-kn} \tag{24.444}$$

$$h_n = \frac{1}{N}\sum_{k=0}^{N-1} H_k W_N^{kn} \tag{24.445}$$

The DFT is generally a set of complex numbers. If we have a real signal (which is the case for speech samples), then, since the h_n are real, when $k = 0$, the exponent in Equation 24.441 becomes 0, making the exponential term 1 for all the elements of the summation. Therefore, H_0 becomes real. This term is called the DC term. Also, when N is even (which is usually the case with DFT implementations), then for $k = \frac{N}{2}$, the exponential term of the summation may be written as,

$$\begin{aligned}
e^{-i\frac{2\pi kn}{N}} &= e^{-i\frac{\cancel{2}\pi(\frac{\cancel{N}}{\cancel{2}})n}{\cancel{N}}} \\
&= e^{-i\pi n} \\
&= \cos(n\pi) + i\sin(n\pi) \\
&= \pm 1
\end{aligned}$$

This means that the value of the DFT for the folding frequency, f_c is also real.

Also, the real-ness of the signal means that,

$$H_k = \overline{H_{N-k}} \quad \forall \;\; 0 < k < \frac{N}{2} \tag{24.446}$$

Since $H_0, H_{\frac{N}{2}} \in \mathbb{R}$ as shown earlier, we may add these two cases to the list in Equation 24.446 so that,

$$H_k = \overline{H_{N-k}} \quad \forall \;\; 0 \le k \le \frac{N}{2} \tag{24.447}$$

where N is even.

Note the similarity of the IDFT to DFT. In a practical sense, with slight modifications, the DFT may be used to compute the IDFT. This is done in practice.

Therefore, there is only need for the implementation of one side of the algorithm, the DFT. Later, we will see that there are efficient techniques for computing the DFT. Fast Fourier Transform is one such algorithm which will be discussed later.

24.10.2 Periodicity

Now, consider H_{k+N},

$$\begin{aligned} H_{k+N} &= \sum_{n=0}^{N-1} h_n e^{-i\frac{2\pi(k+N)n}{N}} \\ &= \sum_{n=0}^{N-1} h_n e^{-i\frac{2\pi kn}{N}} e^{-i\frac{2\pi \cancel{N}n}{\cancel{N}}} \\ &= \sum_{n=0}^{N-1} h_n e^{-i\frac{2\pi kn}{N}} \\ &= H_k \end{aligned} \tag{24.448}$$

Equation 24.448 suggests that the set of H_k is periodic with period N.

In the case where we have a real signal (such as speech), then by only knowing the first $\frac{N}{2}+1$ elements, we will know the information for any index since the elements from $\frac{N}{2}+1$ to $N-1$ are complex conjugates and easily determined by the first $\frac{N}{2}+1$ elements and the indices for N and higher are just periodically related to the first N numbers.

To recapitulate, H_0 corresponds to the DC level, $H_{\frac{N}{2}}$ corresponds to f_c. Indices $0 < k < \frac{N}{2}-1$ corresponds to $0 < f < f_c$ and $\frac{N}{2}+1 < k < N$ correspond to $-f_c < f < 0$.

24.10.3 Plancherel and Parseval's Theorem

Following the example of Parseval's Theorem for the Complex Fourier Series (Section 24.6.2) and the Complex Fourier Transform (Section 24.9.7), it may easily be shown that for two sampled signals, g_n and h_n,

$$\sum_{n=0}^{N-1} g_n \overline{h_n} = \frac{1}{N} \sum_{k=0}^{N-1} G_k \overline{H_k} \tag{24.449}$$